Hand Out - I

Three Examples to compute generalized eigenvectors.

Example

HO1.1

$$A = \begin{pmatrix} 10 & 36 \\ -1 & -2 \end{pmatrix}$$

Char poly of A is

$$(\lambda - 10)(\lambda + 2) + 36$$
$$= \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$$

Two eigenvalues at 4, 4.

$$(A - 4I) = \begin{pmatrix} 6 & 36 \\ -1 & -6 \end{pmatrix}$$

$$(A - 4I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Find v: $(A - 4I)^2 v = 0$, $(A - 4I)v \neq 0$.

HO 1.2

If such a v exists I claim that the vectors

$$u = (A-4I)v \ \& \ v$$

are two generalized pair of eigenvectors.

why?

because

$$(A-4I)u = (A-4I)^2 v = 0$$

Hence u is an eigenvector

$$\boxed{Au = 4u}$$

Moreover

$$u = (A-4I)v \Rightarrow \boxed{Av = 4v + u}.$$

Q: Can I find such a v.

Ans: choose $v = \begin{pmatrix} a \\ b \end{pmatrix}$.

HO1.3

$(A-4I)v \neq 0 \Rightarrow 6a+36b \neq 0$

$$\Rightarrow \boxed{a \neq -6b}$$

$(A-4I)^2 v$ is always 0. So any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ where $a \neq -6b$ would suffice. For example $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ would do.

If $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we have

$$u = (A-4I)v = \begin{pmatrix} 6 & 36 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 42 \\ -7 \end{pmatrix}$$

It is easy to see that $u = \begin{pmatrix} 42 \\ -7 \end{pmatrix}$ is an eigenvector.

(H01.4)

Example:

$$A = \begin{pmatrix} 104 & -215 & -344 \\ 48 & 336 & 96 \\ 56 & 70 & 388 \end{pmatrix}$$

Three eigenvalues at 276, 276, 276.

$$\boxed{A - 276I = \begin{pmatrix} -172 & -215 & -344 \\ 48 & 60 & 96 \\ 56 & 70 & 112 \end{pmatrix}}$$

$$\boxed{(A-276I)^2 = 0}$$

Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We want to find v:

$(A-276I)v \neq 0$

$(A-276I)^2 v = 0$. ← always true

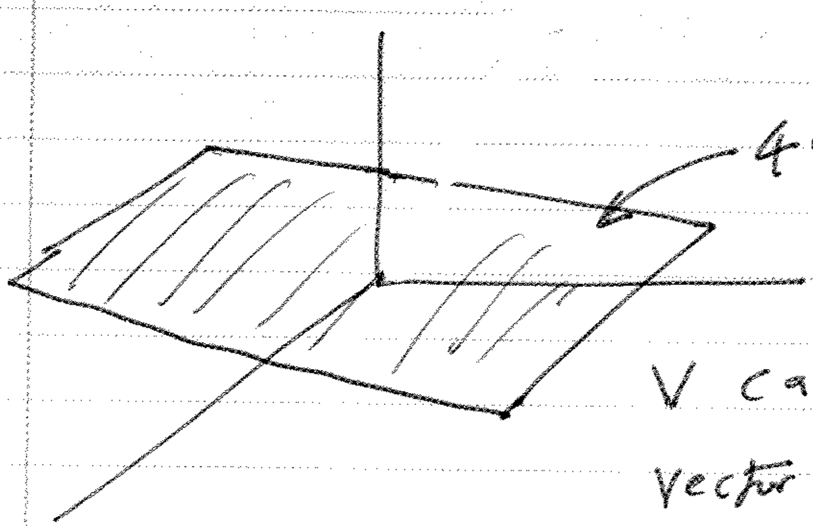
H01.5

$(A - 276I)V \neq 0$

$$\Rightarrow \begin{pmatrix} -172 & -215 & -344 \\ 48 & 60 & 96 \\ 56 & 70 & 112 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 5 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0 \quad 0$$

only one independent equation.



$4a + 5b + 8c = 0$

V can be any vector outside this plane.

H01.6

The plane

$$4a+5b+8c=0$$

contains the eigenvectors of A.

I am looking for a vector which is not an eigenvector, and hence outside the plane.

Let

$$v=\begin{pmatrix}1\\1\\1\end{pmatrix}$$ verify that this vector is outside the plane.

using v, define

$$u=(A-276I)\,v=\begin{pmatrix}-731\\204\\238\end{pmatrix}$$

HO1·7

verify that this u is on the plane and that u is an eigenvector.

Thus $V \longrightarrow u$ form a pair of generalized chain of eigenvectors out of which u is actually an eigenvector.

$$\boxed{\begin{aligned} Au &= 276u. \\ AV &= 276V + u. \end{aligned}}$$

Q: Where is the other eigen-vector.

Ans: It is any vector l.i. with respect to u and belongs to the plane $4a+5b+8c=0$.

(H01·8)

The vector $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ is such a vector.

Conclusion.

$$\underset{v_2}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_1}{\begin{pmatrix} -731 \\ 204 \\ 238 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}.$$

are three independent generalized eigenvector of A.

v_1 & v_3 are eigenvectors.

v_2 is a gen. "

v_1 & v_2 forms a pair.

Caution: V_1 & V_2 are related to each other. If you choose v_1 first, it will be hard to choose v_2. However once you choose v_2,

$$v_1 = (A - \lambda I)\, v_2$$

(H01.9)

Finally if we stack up the vectors V_1, V_2, V_3 in the form of a matrix we get

$$P = \begin{pmatrix} -731 & 1 & -2 \\ 204 & 1 & 0 \\ 238 & 1 & 1 \end{pmatrix}$$

Verify that $P^{-1}AP =$

$$\left(\begin{array}{cc|c} 276 & 1 & 0 \\ 0 & 276 & 0 \\ \hline 0 & 0 & 276 \end{array}\right)$$

This is that famous jordan canonical form.

— x —

Example 3

H01·10

B =

$$\begin{pmatrix} -7408 & 12389 & 11033 & -14387 & -12197 \\ -7164 & -27729 & -11997 & 14571 & 12384 \\ 2659 & 10231 & -2588 & -11608 & -10963 \\ -8851 & -16432 & -12289 & -2531 & 10072 \\ 5953 & 10699 & 7906 & -4435 & -17308 \end{pmatrix}$$

```
>> [v1 v2]=jordan(B)

v1 =

  1.0e+006 *

   -0.0025   -0.0000   -8.0158    0.0014    0.0000
    0.0021    0.0000    7.4689   -0.0010   -0.0000
   -0.0016   -0.0000   -2.5184   -0.0008    0.0000
   -0.0031    0.0000    3.7704    0.0004   -0.0000
    0.0029   -0.0000   -0.5756   -0.0013    0.0000
```

These numbers are not very useful.

Eigenvalue −14391 repeated twice

v2 =

$$\begin{pmatrix} -14391 & 1 & 0 & 0 & 0 \\ 0 & -14391 & 0 & 0 & 0 \\ 0 & 0 & -9594 & 1 & 0 \\ 0 & 0 & 0 & -9594 & 1 \\ 0 & 0 & 0 & 0 & -9594 \end{pmatrix}$$

Eigenvalue −9594 repeated three times

```
>> I=[1 0 0 0 0;0 1 0 0 0; 0 0 1 0 0 ; 0 0 0 1 0;0 0 0 0 1]
```

I =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

H01·11

```
>> B1=B+14391*I
```

$B1 = (B - \lambda I)$

```
B1 =
```

$$\begin{pmatrix} 6983 & 12389 & 11033 & -14387 & -12197 \\ -7164 & -13338 & -11997 & 14571 & 12384 \\ 2659 & 10231 & 11803 & -11608 & -10963 \\ -8851 & -16432 & -12289 & 11860 & 10072 \\ 5953 & 10699 & 7906 & -4435 & -2917 \end{pmatrix}$$

```
>> B2=B1*B1
```

$B2 = (B - \lambda I)^2$

```
B2 =
```

$$\begin{pmatrix} 44074836 & 140058009 & 139007466 & -164551491 & -162028269 \\ -41618772 & -140528115 & -141780132 & 165870666 & 164359611 \\ 14136759 & 90687285 & 101883483 & -115238331 & -120064113 \\ -21778380 & -103336974 & -111683754 & 126549657 & 129259962 \\ 7833501 & 53601678 & 62077977 & -61185735 & -62946234 \end{pmatrix}$$

```
>> rank(B1)

ans =

     4

>> rank(B2)

ans =

     3

>> rank(B1*B1*B1)

ans =

     3
```

HO1·12

```
>> null(B1,'r')

ans =
```

$$\begin{pmatrix} -0.8603 \\ 0.7151 \\ -0.5534 \\ -1.0740 \\ 1.0000 \end{pmatrix}$$

Null space of B1 is spanned by one vector which is the an eigenvector of B for the eigenvalue −14,391.

```
>> t1=null(B1,'r')

t1 =

   -0.8603
    0.7151
   -0.5534
   -1.0740
    1.0000

>> t2=null(B2,'r')

t2 =
```

$$\begin{pmatrix} -4.7857 & -6.0000 \\ 7.7143 & 9.0000 \\ -5.0714 & -6.0000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{pmatrix}$$

The null space of B2 is spanned by the two column vectors.

HO1·13

```
>> t=t2(:,2)

t =

    -6
     9
    -6
     0
     1

>> s=B1*t

s =

      -8792
       7308
      -5656
     -10976
      10220

>> ss=B1*s

ss =

     0
     0
     0
     0
     0

>>
```

I picked the second column.
Note that the vector 't' has the property that

$$(B-\lambda I)^2 t = 0$$

$$s = (B-\lambda I)t \neq 0$$

Thus s & t form a pair of generalized eigenvectors of B.
's' is actually an eigenvector which is what I verify here

$$(B-\lambda I)s = 0 .$$

HO1·14

Conclusion:

This was 't'

This was 's'.

$$v_2 = \begin{pmatrix} -6 \\ 9 \\ -6 \\ 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -8792 \\ 7308 \\ -5656 \\ -10{,}976 \\ 10{,}220 \end{pmatrix}$$

form a chain of generalized eigenvectors of B for the eigenvalue -14391

In fact

$$\boxed{\begin{aligned} (B + 14391\, I)\, v_1 &= 0 \\ (B + 14391\, I)\, v_2 &= v_1 \end{aligned}}$$

or

$$\boxed{\begin{aligned} B v_1 &= -14391\, v_1 \\ B v_2 &= -14391\, v_2 + v_1 \end{aligned}}$$

HO1·15

Repeating our calculations for the other eigenvalue $\lambda = -9{,}594$

```
>> B1=B+9594*I

B1 =

    2186    12389    11033   -14387   -12197
   -7164   -18135   -11997    14571    12384
    2659    10231     7006   -11608   -10963
   -8851   -16432   -12289     7063    10072
    5953    10699     7906    -4435    -7714
```

$\leftarrow (B - \lambda I) = B1$

```
>> B2=B1*B1

B2 =

     91143   21197943   33156864  -26522613  -45010251
  27112644   10447866  -26680914   26076492   45547515
 -11373687   -7468929   11656710   -3871179  -14885091
  63138114   54311634    6216912   35776026   32629194
 -49279581  -49044528  -13772187  -18636345  -11949327
```

$\leftarrow (B - \lambda I)^2$

```
>> B3=B1*B1*B1

B3 =

  1.0e+011 *

 -0.9669  -0.8981  -0.5092  -0.6503  -0.2202
 -0.4618  -0.6772   0.2651   0.5405   0.0249
  0.0529   0.1816  -0.2432  -0.4183  -0.0573
 -3.5695  -3.7789  -0.9310  -0.8118  -0.5702
  3.0083   3.1638   0.8275   0.7559   0.4915
```

$\leftarrow (B - \lambda I)^3$

```
>> rank(B1)

ans =

     4

>> rank(B2)

ans =

     3

>> rank(B3)
```

Notice that the rank drops.

H01·16

```
ans =

     2

>> rank(B3*B1)

ans =

     2

>> t3=null(B3,'r')

t3 =

   -2.4286   -3.8571   -0.7143
    2.0476    3.4286    0.5238
    1.0000         0         0
         0    1.0000         0
         0         0    1.0000

>> t=t3(:,1)

t =

   -2.4286
    2.0476
    1.0000
         0
         0

>> s=B1*t

s =

  1.0e+004 *

    3.1092
   -3.1732
    2.1498
   -2.4440
    1.5356

>> ss=B1*s

ss =
```

$(B-\lambda I)^4$

We stop when the rank does not drop any more.

Columns are the null space of $(B-\lambda I)^3$.

t is any vector from the null space of $(B-\lambda I)^3$.

By definition.

$(B-\lambda I)^3 t = 0$

We define

$s = (B-\lambda I) t$

$ss = (B-\lambda I)^2 t$

H01·17

```
1.0e+007 *

 7.6341
-7.1133
 2.3985
-3.5909
 0.5482

>> sss=B1*ss

sss =

1.0e-003 *

 0.5493
-0.7782
 0.4120
-0.6714
 0.4730
```

We verify that $sss = (B-\lambda I)^3 t$ is numerically zero.

Note that t is a vector such that $(B-\lambda I)^3 t = 0$ & $(B-\lambda I)^2 t \neq 0$.

Since we pick 't' randomly from the null space of $(B-\lambda I)^3$, we need to verify this condition. The good news is that almost any vector (but not every vector) in the null space would satisfy the condition.

H 01·18

Conclusion:

The vectors ss, s, t form a chain of generalized eigenvectors of B w.r.t. the eigenvalue $\lambda = -9594$.

$$\boxed{\begin{aligned} &B1 \cdot ss = 0 \Rightarrow (B+9594I)ss = 0 \\ &(B+9594I)s = ss. \\ &(B+9594I)t = s. \end{aligned}}$$

or

'ss' is an eigenvector.

$$\boxed{\begin{aligned} &B \cdot ss = -9594\, ss \\ &B \cdot s = -9594\, s + ss \\ &B \cdot t = -9594\, t + s \end{aligned}}$$

H01·19

We have found a set of 5 generalized eigenvectors for our original matrix B in page H0 1·10.

$$\underbrace{\begin{pmatrix} -8792 \\ 7308 \\ -5656 \\ -10976 \\ 10220 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} -6 \\ 9 \\ -6 \\ 0 \\ 1 \end{pmatrix}}_{v_2} \leftarrow \lambda = -14391$$

$\lambda = -9,594$.

eigenvectors

$$\underbrace{\begin{pmatrix} 7.63 \\ -7.11 \\ 2.40 \\ -3.59 \\ .55 \end{pmatrix}}_{v_3} \times 10^7, \underbrace{\begin{pmatrix} 3.11 \\ -3.17 \\ 2.15 \\ -2.44 \\ 1.53 \end{pmatrix}}_{v_4} \times 10^4, \underbrace{\begin{pmatrix} -2.43 \\ 2.65 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_5}$$